WHAT IS NETWORK SYNTHESIS?

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Introduction

In the investigations of the theory of network synthesis upon which the present discussions are based, considerable time and effort were devoted to a review of the foundations of this theory. The necessity for such a critical study was indicated by the growing interest in recent years in the problem of time domain synthesis. In pursuit of this problem it was found that the foundations and postulates used in frequency domain synthesis, while still correctly applicable, were potentially insufficient to characterize adequately the situations met in the process of time domain synthesis.

For a number of years we have known about theorems having to do with the existence of positive real functions. These theorems, due primarily to O. Brune, state the necessary and sufficient conditions for a given function to be positive real; that is, to be the driving point impedance or admittance of a passive network. Similar theorems for transfer functions were not at hand. A significant result of our recent investigations is the production of a theorem regarding the existence of transfer functions. The theorem states the necessary and sufficient condition for a given function to be "transfer," and provides the means for the construction of arbitrary transfer functions. The existence conditions are given in terms of Stieltjes integrals, and the theorem relates both to continuous as well as to discrete networks.

We have known for a long time that the lattice structure is the most general symmetrical two terminal-pair network. By means of the new existence theorem for transfer functions we are able to demonstrate the complete generality of the lattice, not only for discrete networks, but for distributed and mixed networks as well. The integral representation of transfer functions yields at once the branch impedances or admittances of the corresponding lattice.

Regarding the correlation of a transfer function in the frequency domain with its mate in the time domain, the Laplace transform theory provides the well known formal relationship. Specifically, if $f(t)$ be a real time function which:

a) is identically zero for $t < 0$,
b) is single valued and bounded for $0 < t < \infty$,
c) has a denumerable set of points of finite discontinuity and a denumerable set of points of impulse behavior, each with finite area,

then its Laplace transform, say F(s), is necessarily a transfer function of the most general type. The converse theorem is also true. That is to say, the inverse Laplace transform of a transfer function is necessarily a bounded function with the same characteristics as those of f(t) defined above.

From a synthesis point of view, these theorems fall short for several reasons. For example, we are for obvious reasons interested in knowing whether and under what conditions a resulting transfer function will be rational. An important result of our recent investigations states the necessary and sufficient conditions for a transfer function to be rational and provides the means for the construction of such functions.

The Laplace transformation is likewise inadequate in providing the lucid and computationally effective link between time and frequency domain functions which is so vitally needed in dealing with synthesis problems. In this regard we have found it useful to express both f(t) and F(s) in terms of a single real, bounded parameter $U(\gamma, \lambda)$ defined along an appropriately chosen path in the s-plane. This parametric representation for the time and frequency domain functions has two important advantages:

a) It subordinates the function F(s) to $U(\gamma, \lambda)$ as being the frequency domain counterpart of f(t) and thus eliminates (because of the real character of U) the traditionally annoying questions attendant upon the complex character of F(s).
b) It makes possible the simultaneous computation and control of errors in time and frequency domains when an approximation process is involved.

The latter process is enhanced through a remarkably simple and computationally convenient relationship between the functions U and f which avoids the necessity of carrying out integrations altogether.

Foremost in the line of practical problems for which these newly developed methods have proved indispensable is that involving the transmission of narrow pulses. Suppression of the familiar phenomenon of "ringing" which is ever-present in connection with problems of this sort, is accomplished through control of corresponding mathematical convergence phenomena.

Coupled with the development of better techniques for accomplishing such ends is, of course, the desire to do so with good if not optimum economy as measured say by the degree of the resulting rational function F(s). The latter part of this paper will provide an illustration for some of these ideas.

In spite of, or perhaps because of the critical evaluation and extension of our available synthesis tools which was provoked by such investigations as mentioned above, there remain some rather elementary questions that have not yet been satisfactorily answered. The most intriguing of these reads: "How shall we define what is meant by network synthesis?" That this question is neither a simple nor a trivial one, will be shown by the immediately following discussion.

Part I

Difficulties Encountered in Time Domain Synthesis

A natural source to which one may turn in an attempt to find a definition for the term "Network Synthesis" is had through observing the fundamental objectives which are to be achieved by this aspect of circuit theory in contrast with those which are sought in network analysis. In this regard we present here the definitions of a high authority* on the subject:

"In network analysis we are primarily concerned with giving all the parameters which will completely determine the electrical behavior of a circuit with n terminal-pairs, if we put any variable voltage (or current) across (or through) one pair or several pairs and observe other pairs."

"When we know the electrical characteristics of an n terminal-pair network, network synthesis is the art of reproducing it with as high an approximation as possible by means of a finite number of elements such as, for example, resistances, capacitances, and inductances," ..."In the case where we are dealing with the known linear problem of network analysis and synthesis, there are other elements which are admissible in our construction as multipliers."

Definitions given by numerous other specialists may differ in detail but agree in objective principle with the above. However, reviewing the present state of the art of network synthesis, we find some disagreement in the establishment of the final goal in network synthesis and in the demarcation of its scope. Two main trends of thought are contained in the following tendencies shown in recent contributions. These may conveniently be described as:

Objective I. The establishment of pertinent theories tending to produce the so-called "system function" which contains the analytical characterization of a required network having a prescribed response under the action of a set of driving forces, constraints, and measures of approximation.

Objective II. The establishment of a constructive process, the so-called "synthesis procedure" for finding the geometrical configuration and element value distribution for one or several networks corresponding to a given system function.

Some writers, most of them mathematically inclined, claim directly or indirectly that the basic purpose of the theory of network synthesis resides in the objective I above. For them, a problem of synthesis ends when the corresponding well characterized system function is found. They consider the accomplishment of objective II to be a mere technicality.

Other writers, however, insist that the main objective of network synthesis basically resides in the objective II. They base this view on two facts:

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*The author is indebted to Prof. Norbert Wiener for the preparation of these definitions.

a) The term "network synthesis" implies, by its direct meaning, the constructive existence of the network rather than its mathematical image.

b) The production of the geometry of the network has proved to be a very difficult problem in its own right, and it is, therefore, far from being a mere technicality. Consequently, they argue, network synthesis can more reasonably be described by the second rather than by the first of the two objectives given above. The supporters of the second objective consider the first one as being a part of "network analysis."

Recent developments clearly indicate that neither of the above objectives is potentially sufficient by itself to characterize the domain of network synthesis. We have learned that both objectives contribute formally with equal weight to such a characterization, and it is, therefore, meaningless to separate them in defining the aims of circuit synthesis. We have further learned that both objectives together are still potentially insufficient for the final determination of a geometrical distribution of elements constituting a satisfactory solution. The purpose of the immediately following discussion is to illustrate this situation by means of pertinent examples.

We begin by introducing the postulate of "finiteness." This postulate is nothing more than an expression of the physical fact that we cannot construct infinite networks. We will say that a network is <u>finite</u> if: a) The discrete part, if any, is formed by a finite total number of concentrated elements each having a finite value. b) The distributed part, if any, is formed by a continuous distribution of (differential) elements such that the network has finite total dimensions with a finite number of terminals.

Now, the finiteness condition leads us to say that "a synthesis problem has a solution if, and only if, the representative network is finite in the above sense." Consequently, infinite network solutions are no solutions at all.

In spite of its natural simplicity, this postulation has a serious impact upon the theory of network synthesis. For a long time we have known that very many problems, particularly in time domain synthesis, require for their solution a transcendental transfer function, which cannot be constructed with finite discrete networks. Consequently it may appear, under the postulate of finiteness, that the theory of network synthesis has a restricted field of application.

A basic task of this and other researches in network synthesis is the investigation of the possibilities and limitations of this theory under the postulate of finiteness. Experience tells us that the demarcation of the field of network synthesis is not so restricted if one accepts "approximate" solutions inside prescribed tolerances instead of asking for "exact" solutions.

From the very beginning of the development of network synthesis we

have been extracting approximate finite solutions from "infinite" ones by terminating the latter at convenient points. This method has proved to be very useful in some cases and very treacherous in many others.

The classical approach to this method can be described as follows: We have known since Brune that when the corresponding system function belongs to a certain class of rational functions (a class to be characterized in some other part of this paper) then the corresponding network is necessarily finite, passive, and linear. If

$$F^*(s) = F_n(s) = \frac{P_n(s)}{Q_n(s)} \tag{1}$$

be such a rational function, and if a family of such functions for different integers n fulfills the condition

$$\left\{ F(s) - \frac{P_n(s)}{Q_n(s)} \right\}_{n \to \infty} \longrightarrow \quad 0 \text{ uniformly,} \tag{2}$$

wherein F(s) is the desired (transcendental) system function, then the function $F^*(s)$ is regarded as an appropriate rational approximant.

The trouble with this simple argument is that it may be right or completely wrong depending on whether we choose the above objective I or II or both as a definition of network synthesis. This is a peculiar and illuminating situation which makes us realize that, in spite of almost three decades of productive effort, we still do not clearly understand what network synthesis really is, or should be. We will illustrate these remarks by means of a simple example.

Let g(t) be a real, piecewize continuous, single valued and bounded function of t, such that

$$g(t) = \begin{cases} \equiv 0 \text{ for } t < 0, \\ \not\equiv 0 \text{ for } t > 0. \end{cases} \tag{3}$$

A proposed particular synthesis problem is the following: We wish to find a discrete passive and finite two terminal-pair network whose response h(t) to the excitation g(t) is a simple delay of this function, thus

$$h(t) = \begin{cases} \equiv 0 \text{ for } t < 1, \\ \equiv g(t-1) \text{ for } t > 1. \end{cases} \tag{4}$$

The elementary theory of Laplace transforms tells us that the required system function of the network is uniquely given by

$$F(s) = e^{-s}: \quad s = \sigma + j\omega. \tag{5}$$

Evidently, when we say that e^{-s} is the unique solution for F(s),

we have completely complied with objective I above. However, we could not rationally describe this first step alone as a complete synthesis procedure because we have not given:

1.) any direct assurance for the physical existence of the desired network, or

2.) any direct assurance that there exists a finite network whose response approaches h(t) as described above for every possible excitation function g(t) satisfying Eq. 3.

Let us consider the objective II. This objective deals with the realization of a network having the system function e^{-s} independently of the function g(t). We know that we can find a sequence of partial networks (delay networks) which successively accomplish the realization of e^{-s} with tolerances that become arbitrarily small. A typical family of such networks is obtained, for example, through using the Pade' table method of determining a set of rational approximants for the function e^{-s}.

To find these rational approximants for a function F(s) we begin with its Taylor expansion

$$F(s) = C_o + C_1 s + C_2 s^2 + \dots + C_n s^n + \dots , \tag{6}$$

and consider two polynomials

$$\begin{aligned} P_\nu(s) &= \alpha_o + \alpha_1 s + \alpha_2 s^2 + \dots + \alpha_\nu s^\nu \\ Q_\mu(s) &= \beta_o + \beta_1 s + \beta_2 s^2 + \dots + \beta_\mu s^\mu \end{aligned} \tag{7}$$

of degree ν and μ respectively. Then the Pade' approximants are defined as that set for which

$$F(s) - \frac{P_\nu(s)}{Q_\mu(s)} = 0 + 0 + \dots + 0 + C_{\nu+\mu+1} s^{\nu+\mu+1} + C_{\nu+\mu+2} s^{\nu+\mu+2} + \dots \tag{8}$$

for any combination of the integers ν and μ. The coefficients α and β are uniquely determined from the equations

$$\begin{aligned} C_o\beta_o &= \alpha_o \\ C_1\beta_o + C_o\beta_1 &= \alpha_1 \\ C_2\beta_o + C_1\beta_1 + C_o\beta_2 &= \alpha_2 \\ &- - - - - - - - - - - - - \\ C_\nu\beta_o + C_{\nu-1}\beta_1 + \dots + C_{\nu-\mu}\beta_\mu &= \alpha_\nu \\ C_{\nu+1}\beta_o + C_\nu\beta_1 + \dots + C_{\nu+1-\mu}\beta_\mu &= 0 \\ &- - - - - - - - - - - - - - - - - \\ C_{\nu+\mu}\beta_o + C_{\nu+\mu-1}\beta_1 + \dots + C_\nu\beta_\mu &= 0. \end{aligned} \tag{9}$$

Through use of the confluent hypergeometric functions it is possible to obtain explicit expressions for P_ν and Q_μ directly. Specifically, for $\nu = \mu$ (which leads to the approximants located in the principal diagonal of the Pade' table), we find for the function $F(s) = e^{-s}$ the polynomials

$$P_\mu(s) = 1 - \frac{1}{2}s + \frac{\mu(\mu-1)}{2\mu(2\mu-1)} \cdot \frac{s^2}{2!} - \dots + (-1)^\mu \frac{\mu(\mu-1) \dots 2 \cdot 1}{2\mu(2\mu-1) \dots (\mu+1)} \cdot \frac{s^\mu}{\mu!},$$

$$Q_\mu(s) = 1 + \frac{1}{2}s + \frac{\mu(\mu-1)}{2\mu(2\mu-1)} \cdot \frac{s^2}{2!} + \dots + \frac{\mu(\mu-1) \dots 2 \cdot 1}{2\mu(2\mu-1) \dots (\mu+1)} \cdot \frac{s^\mu}{\mu!} \quad . \tag{10}$$

Defining the set of rational approximate system functions as

$$F(\overset{*}{s}) = F_\mu(s) = \frac{P_\mu(s)}{Q_\mu(s)}, \tag{11}$$

and regarding each of these as the transfer impedance of a constant resistance lattice, we obtain the sequence of networks illustrated in Fig. 1.

According to objective II above, we have completely solved the problem of synthesis, for we have provided network configurations and element values for a sequence of partial, finite networks whose transfer functions $F_\mu(s)$ approach uniformly towards e^{-s} as close as we wish by making μ sufficiently large.

However, it can be denied that a synthesis problem really has been solved, for we shall show that the above solution does or does not make sense practically according to the nature of the excitation function $g(t)$. If this is true, then the objective II alone is potentially insufficient to define what we mean by "network synthesis."

In order to show the effect of the nature of $g(t)$, let us consider one of the partial networks constructed for a fixed value of μ, say $\mu = 6$. We will study its response when it is excited successively by each of the following strategically selected functions.

$$\text{Case A} \quad g_1(t) = \begin{cases} \equiv 0 & \text{for } t < 0, \\ = t/t_o & \text{for } 0 < t < t_o, \\ = 1 & \text{for } t > t_o, \end{cases} \tag{12}$$

$$\text{Case B} \quad g_2(t) = \begin{cases} \equiv 0 & \text{for } t < 0, \\ = 1 & \text{for } t > 0, \end{cases} \tag{13}$$

$$\text{Case C} \quad g_3(t) = u_o(t). \tag{14}$$

The last two of these are the familiar unit step and the unit impulse, both occuring at $t = 0$. Fig. 2 illustrates these excitation functions

as well as the corresponding desired responses.

Since the networks of Fig. 1 have system functions that approximate e^{-s}, we expect that their responses which we may call $h_1^*(t)$; $h_2^*(t)$ and $h_3^*(t)$, will resemble the delayed functions $g_1(t-1)$, $g_2(t-1)$ and $g_3(t-1)$ respectively. But this is not consistently the case as the plots for $h^*(t)$ in Fig. 3 (computed for $\mu = 6$) show. Thus, inspection of the graph for Case A shows that such a Pade' network behaves beautifully in reproducing the delayed function $g_1(t)$ of Eq. 12, even for the moderate value of the integer $\mu = 6$.

The graph for Case B shows that the same network does not reproduce the function $g_2(t-1)$ very well for $0 < t < 1$, although its response thereafter is quite acceptable. A spike of unit height appears at $t = 0$ for all values of μ, a spike which is not present in $g_2(t-1)$. We note, however, the evident tendency of this network to produce the delay associated with e^{-s}.

The graph for Case C shows that the same network completely fails to reproduce $g_3(t-1)$, at least for $\mu = 6$. Rapid oscillations take place in the vicinity of $t = 0+$, and the tendency of producing a delay seems to be completely absent.

One may, of course, argue that the value of $\mu = 6$ is small for such cases as B and C, and so one may think that by increasing μ to a sufficiently high finite number, the situation will necessarily improve. So it does for Case B, as is shown in the plots of Fig. 4 computed for $\mu = 20$. Here one observes that the spurious oscillations within the interval $0 < t < 1$ tend to zero as μ increases except for the ever-present spike at $t = 0$ which persists for all values of μ+.

The increase in μ is of no help at all in the Case C, and in fact the situation deteriorates rapidly with an increase in μ. The amplitude of the spurious oscillation increases with an increase in μ while its period decreases. In other words, the function $h_3^*(t)$ diverges by oscillation at every point in the interval $0 < t < 1$ and becomes identically zero everywhere else as $\mu \to \infty$; and this true in spite of the fact that the corresponding frequency domain function $F^*(s)$ <u>converges uniformly toward e^{-s} throughout the entire finite s-plane!</u>

The reader should not conclude that all possible methods of rational approximation produce this phenomenon of complete divergence. The following part of this paper will present one form of synthesis procedure which is free from such drawbacks.

Part II

A Possible Procedure for Successful Time Domain Synthesis.

Failure on the part of a solution to converge properly appears quite often in time domain synthesis. It is due to the following basic causes:

a) The nature of the excitation, particularly if it contains discon-

+ The removal of this spike can easily be accomplished through terminating the networks for z_a and z_b in one-ohm resistances.

tinuities or impulses.

b) The method used in the rational approximation of the system function.

c) The type of approximation criterion which is used, such as the mean square error or the maximal convergence criteria, etc.

Examples of the type of convergence that may result on account of (a) and (b) are shown above. Examples of phenomena of convergence due to the effect of (c) are classically illustrated by the spurious oscillations present in the Gibbs phenomenon which appears at points of discontinuity in the excitation function when the mean square error criterion is used. No matter how much we diminish the error throughout the continuous part of the function, the spurious oscillations appear at the points of discontinuity with an intensity that depends only upon the size of the jump.

Particularly intense phenomena of convergence appear in connection with the transmission of narrow time pulses through almost all finite transmission facilities. This phenomenon is familiarly known as "ringing." Its elimination or suppression has long been considered as one of the most difficult problems in network synthesis. The purpose in the immediately following discussion is to give some specific results which have been found in connection with the study of pulse transmission, and particularly to present a new approach to this problem and compare it with the older procedures described above.

Let us consider the transmission of impulses. The present approach contrasts with the one used in Part I above as follows:

In Part I we started from the delay function e^{-s} directly and expanded it into a rational function. This rational expansion of e^{-s} is not unique. There are an infinite number of such expansions. The Pade' expansion is one of this multitude of possibilities, and the worst attribute of any one of them is that we do not know how to control the phenomenon of ringing.

In the proposed approach we shall forget about e^{-s} explicitly, and instead prescribe the shape of output pulse that we are willing to accept. Fig. 5 illustrates the essence of this attitude. The input is a unit impulse. The desired output $f(t)$ is realistically set down as a pulse of finite height and width (the exact shape being of secondary interest at this point) delayed by t_o seconds. The function $f^*(t)$, differing from $f(t)$ at no point by a larger absolute value than a specified tolerance ε, represents an approximate output which we are willing to accept in place of $f(t)$. $F^*(s)$ is a rational system function whose inverse Laplace transform (except for a possible constant multiplier) is $f^*(t)$, and hence represents the transfer function of the desired network. Given $f(t)$ and ε, our problem is to find $F^*(s)$.

We begin with the following expression which is readily obtained from the Fourier integral representation for a time function $f(t)$, namely

$$f(t) = \frac{1}{\pi}\int_0^{\infty} d\omega \int_{-\infty}^{\infty} f(t-\xi)\cos\omega\xi\, d\xi. \tag{15}$$

Here the second integral represents a convolution of the given time function with a sinusoid. This process becomes especially easy to evaluate if the given f(t) may with sufficient accuracy be approximated by a set of confluent straight lines so that its first derivative is a step function and its second derivative is a finite sum of displaced impulses with values equal to the jumps in the step function. Regarding the convolution integral, it is well known that we can replace one function by its nth derivative and the other one by its nth integral; and also that anything convolved with a displaced unit impulse is that thing similarly displaced.

Thus Eq. 15 may be written

$$f(t) = \frac{-1}{\pi}\int_0^{\infty} d\omega \int_{-\infty}^{\infty} f''(t-\xi)\,\frac{\cos\omega\xi}{\omega^2}\, d\xi \tag{16}$$

where the prime indicates differentiation; and if

$$f''(t-\xi) = \sum_{\nu=1}^{N} d_\nu u_o(t-t_\nu-\xi) \tag{17}$$

in which $d_1 \ldots d_n$ are the discontinuities in the first derivative of f(t) and $u_o(t)$ denotes the unit impulse, then Eq. 15 or 16 evaluates to

$$f(t) = \frac{-1}{\pi}\int_0^{\infty} \sum_{\nu=1}^{N} d_\nu\,\frac{\cos\omega(t-t_\nu)}{\omega^2}\, d\omega\,, \tag{18}$$

or

$$f(t) = -\frac{1}{\pi}\int_0^{\infty} \sum_{\nu=1}^{N} \frac{d_\nu\cos\omega t_\nu}{\omega^2}\cos\omega t\, d\omega - \frac{1}{\pi}\int_0^{\infty} \sum_{\nu=1}^{N} \frac{d_\nu\sin\omega t_\nu}{\omega^2}\cos\omega t\, d\omega. \tag{19}$$

If f(t) is symmetrical about t = 0, then the d_ν-values are equal at $t = \pm t_\nu$, and the second integral in Eq. 19 drops out. If we then regard the $d_1 \ldots d_n$ as pertaining only to the discontinuities in f'(t) for t > 0, we have

$$f(t) = -\frac{2}{\pi}\int_0^\infty \sum_{\nu=1}^{N} \frac{d_\nu \cos\omega t_\nu}{\omega^2} \cos\omega t \, d\omega \, . \tag{20}$$

This function delayed by t_o seconds becomes

$$f(t-t_o) = \frac{-2}{\pi}\int_0^\infty \sum_{\nu=1}^{N} \frac{d_\nu \cos\omega t_\nu}{\omega^2} \cos\omega t_o \cos\omega t \, d\omega, \tag{21}$$

which is a convenient representation for a delayed symmetrical pulse that is formed by straight line segments (as, for example, a triangular or trapezoidal pulse).

Assuming that the pulse has a finite duration and that t_o is large enough to make $f(t-t_o) \equiv 0$ for $t < 0$, we may regard $F(s)$ as the Laplace transform of $f(t-t_o)$ and have according to known methods

$$f(t-t_o) = \frac{2}{\pi}\int_0^\infty u(\omega) \cos\omega t \, d\omega, \tag{22}$$

where $u(\omega)$ is the real part of $F(s)$ along the j-axis of the s-plane. From a comparison of Eqs. 21 and 22 we see in our present example that

$$u(\omega) = -\sum_{\nu=1}^{N} \frac{d_\nu \cos\omega t_\nu}{\omega^2} \cos\omega t_o \, . \tag{23}$$

The quantity

$$\alpha(\omega) = -\sum_{\nu=1}^{N} \frac{d_\nu \cos\omega t_\nu}{\omega^2} \tag{24}$$

is appropriately regarded as an envelope function enclosing the more rapid oscillations given by $\cos\omega t_o$.

It is helpful to recall, according to well known Laplace transform theory, that $\alpha(0)$ equals the area under the pulse, and (by reference to Eqs. 20 and 24) that $2/\pi$ times the area under $\alpha(\omega)$ equals $f(0)$, the pulse height at its center.

As an illustration, consider the trapezoidal pulse of Fig. 6. The formula 24 for the envelope function of the real part of F(s) yields at once

$$\alpha(\omega) = \frac{1}{\omega^2}\left(\frac{3}{2\delta}\cos\omega\frac{\delta}{3} - \frac{3}{2\delta}\cos\omega\delta\right)$$

which may be written

$$\alpha(\omega) = \frac{2\delta}{3} \quad \frac{\cos\omega\frac{\delta}{3} - \cos\omega\delta}{(2\delta\omega/3)^2} \tag{25}$$

The real part $\mathfrak{U}(\omega)$ corresponding to this pulse delayed by $t_o = 10\,\delta/3$ is shown in Fig. 7. Since the envelope has an assymptote that varies inversely as ω^2, the $\mathfrak{U}$-function becomes small rather rapidly. Hence an approximate function $\mathfrak{U}^*(\omega)$ which essentially coincides with $\mathfrak{U}(\omega)$ from $\omega = 0$ to some appropriately chosen point like $\omega_N = N\omega_1$ should yield an acceptable approximation $f^*(t - t_o)$ and lead to a rational system function $F^*(s)$. We shall next compute an approximate upper bound for the error ε thus committed in the time domain, and then describe the construction of $\mathfrak{U}^*(\omega)$ consistent with this error.

The error is evidently given by the expression

$$\varepsilon = f(t) - f^*(t) = \frac{2}{\pi}\int_{\omega_N}^{\infty} \alpha(\omega)\cos\omega t_o \cos\omega t \, d\omega \,. \tag{26}$$

Let us consider separately the contribution to the total value of ε contributed by one lobe of the envelope function, as expressed by the integral

$$I_n = \frac{2}{\pi}\int_{\omega_N}^{\omega_{N+1}} \alpha(\omega)\cos\omega t_o \cos\omega t \, d\omega \,. \tag{27}$$

An approximate value for this integral is had through considering the lobe of $\alpha(\omega)$ within the interval $\omega_n < \omega < \omega_{n+1}$ to be replaceable by a sinusoidal hump having the same maximum. According to the notation indicated in Fig. 7 and the expression 25 for $\alpha(\omega)$, this maximum is given by

$$h_{2N+1} = \frac{8\sqrt{2}\,\delta}{3\pi^2(2N+1)^2} \,. \tag{28}$$

The approximate absolute value thus obtained for the integral 27 may

then be written

$$|I_n| \approx \frac{2}{\pi} h_{2N+1} \int_0^{\omega_1} \sin\frac{\pi}{\omega_1} x \cos(\omega_N + x) t_o \cos(\omega_N + x) t \, dx \, . \tag{29}$$

Since the sine function in the integrand remains positive over the pertinent interval, the integral has a largest value for $t = t_o$, and the remainder of the integrand then becomes

$$\cos^2(\omega_N + x) t_o = \frac{1}{2}\left[1 + \cos 2(\omega_N + x) t_o\right] .$$

Noting that the double frequency term will contribute practically nothing, we have for an approximate upper bound

$$|I_N|_{max} \approx \frac{1}{\pi} h_{2N+1} \int_0^{\omega_1} \sin\frac{\pi}{\omega_1} x \, dx = \frac{2\omega_1 h_{2N+1}}{\pi^2} \, . \tag{30}$$

Substitution of the appropriate values for ω_1 and h_{2n+1} yields

$$|I_N|_{max} \approx \frac{2\sqrt{2}}{\pi^3 (N + \frac{1}{2})^2} \, . \tag{31}$$

The contributions to the error ε in Eq. 26 due to successive lobes in the α-function will not necessarily have the same algebraic sign, and so a somewhat pessimistic upper bound will be had through simply summing the expression 31 as is done in the following relation yielding an estimate of the time domain error for the trapezoidal pulse of Fig. 6

$$\varepsilon = \frac{2\sqrt{2}}{\pi^3} \sum_{n=N}^{\infty} \frac{1}{(n + \frac{1}{2})^2} \, . \tag{32}$$

Numerical evaluation of this expression is facilitated through use of the generalized Riemann zeta function* defined by

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s} \, . \tag{33}$$

Our case above pertains to $a = \frac{1}{2}$, $s = 2$, for which**

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*Whittaker and Watson "Modern Analysis" p. 265
**Jahnke Emde (Dover 1943) p. 269 e.s.

$$\varphi(2, \tfrac{1}{2}) = 3\,\varphi(2) = 4.935.$$

Hence the relation 32 may be written

$$\varepsilon = \frac{2\sqrt{2}}{\pi^3}\left\{4.935 - \sum_{n=0}^{N-1} \frac{1}{(n+\frac{1}{2})^2}\right\}. \tag{34}$$

The following table gives upper bounds for the error computed from this formula

N	ε
1	4.5%
2	3.0%
3	2.3%
4	1.6%

Since the actual maximum error is very probably considerably less than this estimate of an upper bound, it is clear that a very good result may be obtained through cutting the $u(\omega)$ function of Fig. 7 at a reasonably low point on the frequency scale. It remains to show how we can construct an appropriate rational expression for $u^*(\omega)$ thus established.

It is well known that this real part is a quotient of finite polynomials in ω^2, thus

$$u^*(\omega) = \frac{P_n(\omega^2)}{Q_m(\omega^2)} \quad . \tag{35}$$

The zeros of $u^*(\omega)$ in number and location are known from the plot of Fig. 7. Hence the polynomial $P_n(\omega^2)$ is immediately constructible except for an unimportant constant multiplier.

We now write

$$Q_m(\omega^2) = q_m\omega^{2m} + q_{m-1}\omega^{2(m-1)} + \dots + q_1\omega^2 + q_o, \tag{36}$$

and note that the necessary boundedness of $u^*(\omega)$ requires $m > n$. It will suffice to set $m = n+1$. We then write Eq. 35 in the form

$$\frac{P_n(\omega^2)}{u^*(\omega)} = q_m\omega^{2m} + q_{m-1}\omega^{2(m-1)} + \cdots + q_1\omega^2 + q_o \tag{37}$$

and, choosing the $m+1$ ω-values given by

$$\omega = \omega_{\nu} = \frac{\nu\pi}{t_o} \quad ; \quad \nu = 0, 1, 2, \ldots n, \tag{38}$$

which are the points at which $\mathfrak{U}^*(\omega)$ passes through its maxima and minima (very nearly), we obtain a set of simultaneous algebraic equations solvable for the coefficients $q_o, q_1, \ldots q_m$. The solution of those equations presents no computational difficulties. Since their determinant is of the Vandermonde type, the solution can be expressed in closed form.

It may finally be shown that the resulting rational function $\mathfrak{U}^*(\omega)$, from which $F^*(s)$ is readily obtained by Gewertz's or Bode's methods, has the lowest degree consistent with the associated time domain error ε. This is done by regarding the $\mathfrak{U}^*(\omega)$ function of Fig. 7 as formed by a succession of positive and negative humps or spikes, and showing that the removal or suppression of any one spike immediately produces ringing in excess of the error ε. Since the degree of $\mathfrak{U}^*$, and hence that of $F^*(s)$, is directly related to the total number of spikes comprising $\mathfrak{U}^*(\omega)$, the stated minimum degree property is evident.

Further details regarding these latter points as well as more general aspects of the above method exploiting use of the real part of $F(s)$ as the frequency domain counterpart of $f(t)$, rather than $F(s)$ itself, are to extensive to permit discussion at this point. They are being prepared as reports of the M.I.T. Research Laboratory of Electronics.

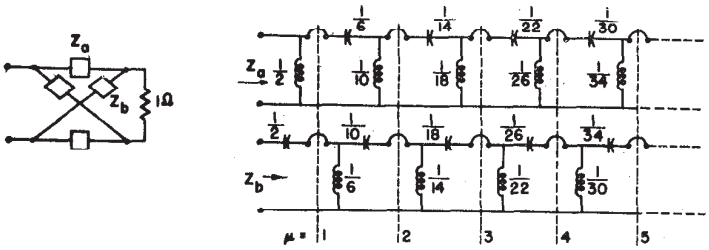

Fig. 1

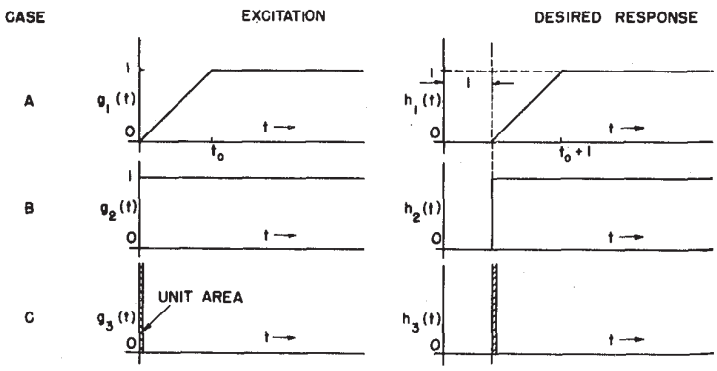

Fig. 2

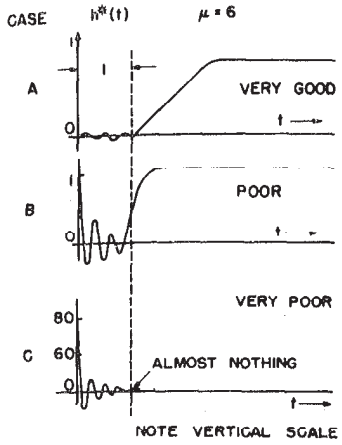

Fig. 3

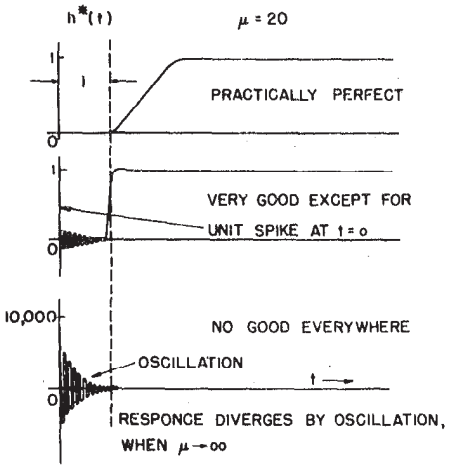

Fig. 4

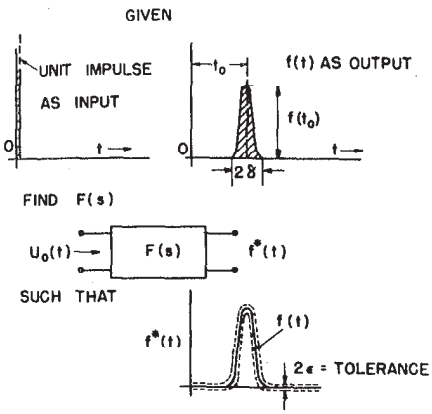

Fig. 5

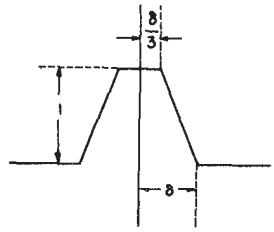

Fig. 6

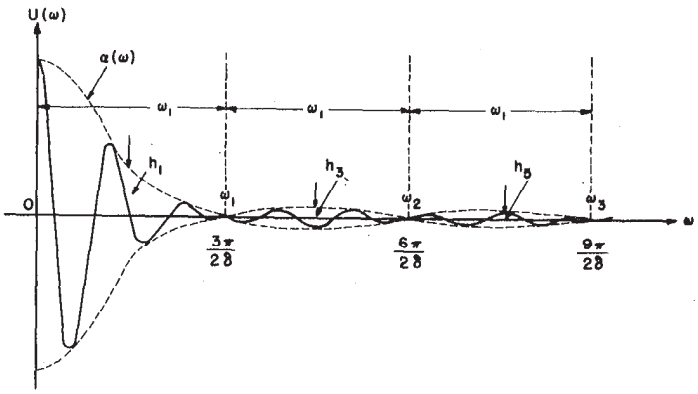

Fig. 7